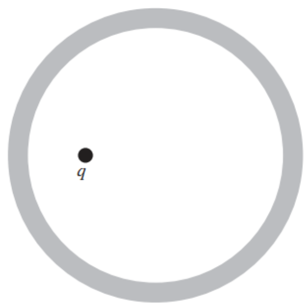
Lecture 13 - Linking $\vec{E}$, ϕ, and ρ

A Puzzle...

Inner-Surface Charge Density

A positive point charge q is located off-center inside a neutral conducting spherical shell. We know from Gauss's law that the total charge on the inner surface of the shell is $-q$. Is the surface charge density negative over the entire inner surface? Or can it be positive on the far side of the inner surface if the point charge q is close enough to the shell so that it attracts enough negative charge to the near side? Justify your answer.
Hint: Think about field lines.



Solution
If there were a location with positive density, then electric field lines would start there, pointing away from it into the spherical cavity. But where could these field lines end? They can't end at infinity, because that's outside the shell. And they can't end at a point in empty space, because that would violate Gauss's law; there would be nonzero flux into a region that contains no charge. They also can't end on the positive point charge q, because the field lines point outward from q. And finally they can't end on the shell, because that would imply a nonzero line integral of $\vec{E}$ (and hence a nonzero potential difference) between two points on the shell. But we know that all points on the conducting shell are at the same potential. Therefore, such a field line (pointing inward from the inner surface) can't exist. So all of the inner surface charge must be negative. Every field line inside the cavity starts at the point charge q and ends on the shell. □

A 30,000 Foot View (From Last Time)

It is important to realize that although we have been focusing specific charge distributions from points, lines, sheets, and spheres, electricity is one of the most prevalent and influential forces that we encounter in our daily lives! This remarkable YouTube video highlights some amazing electricity tricks that you can easily test out for yourself!

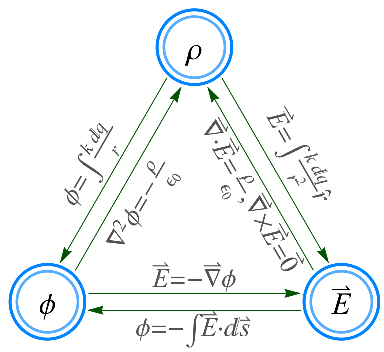
Problems

Complementary Section: The Missing Link

By now, we are familiar with the three fundamental quantities in electrostatics, namely the:

- Electric field ($\vec{E}$) - The electric field $\vec{E}[\vec{r}]$ at a point $\vec{r}$ equals the force per unit charge that a point charge would feel at $\vec{r}$.
- Electric potential (ϕ) - The potential $\phi[\vec{r}]$ equals the amount of work that must be done by an external agent in carrying a unit of positive charge from the reference point (usually infinity) to $\vec{r}$ without any acceleration.
- Charge density (ρ) - The charge distribution everywhere.

These three quantities are connected by the following relations.



The only relation that we are not intimately familiar with is

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{1}$$

which arises from combining $\vec{E} = -\vec{\nabla}\phi$ together with $\frac{\rho}{\epsilon_0} = \vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\vec{\nabla}\phi \equiv -\nabla^2\phi$ where we have defined the <u>*Laplacian*</u>

$$\nabla^2\phi = \frac{\partial\phi}{\partial x^2} + \frac{\partial\phi}{\partial y^2} + \frac{\partial\phi}{\partial z^2} \tag{2}$$

in Cartesian coordinates. Surprisingly, Equation (1) turns out to be one of the most useful relations in the diagram above because the Laplacian has many wonderful properties. For example, as discussed in class Equation (1) in free space ($\rho = 0$) becomes

$$\nabla^2\phi = 0 \tag{3}$$

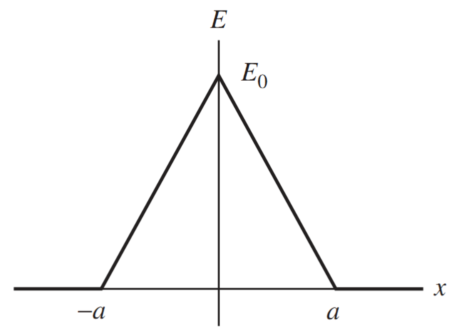
whose solution $\phi[\vec{r}]$ equals the average value of ϕ in a small sphere around $\vec{r}$. This implies that ϕ has no minima or maxima, and therefore that you cannot construct an electrostatic field that will hold a charged particle in stable equilibrium. In more advances physics courses, this will be the primary problem solving tool in many electrostatics problems.

Complementary Section: Two Concentric Shells

Triangular *E*

<u>Example</u>

Find the charge density ρ and potential φ associated with the electric field shown in the figure below. E is independent of y and z. Assume that $\varphi = 0$ at $x = 0$.



Solution

Since E is independent of y and z, we should integrate $\phi = -\int \vec{E}[x] \cdot d\vec{s}$ along the x-axis. Note that because the charge distribution goes out to infinity, we don't set the zero potential at infinity (which would not be well defined), but instead use $x = 0$. At a distance $0 < x < a$,

$$\phi = -\int_0^x E[\tilde{x}]\, d\tilde{x} = -\int_0^x E_0\left(1 - \frac{\tilde{x}}{a}\right) d\tilde{x} = -E_0\left(\tilde{x} - \frac{\tilde{x}^2}{2\,a}\right)_{\tilde{x}=0}^{\tilde{x}=x} = E_0\left(\frac{x^2}{2\,a} - x\right) \tag{4}$$

For distances $x > a$, note that $E[x] = 0$ for $x > a$ so that $\phi[x] = \phi[a]$ for this range. For $x < 0$, we do a similar calculation to Equation (4) above,

$$\phi = -\int_0^x E[\tilde{x}]\, d\tilde{x} = -\int_0^x E_0\left(1 + \frac{\tilde{x}}{a}\right) d\tilde{x} = -E_0\left(\tilde{x} + \frac{\tilde{x}^2}{2\,a}\right)_{\tilde{x}=0}^{\tilde{x}=x} = -E_0\left(\frac{x^2}{2\,a} + x\right) \tag{5}$$

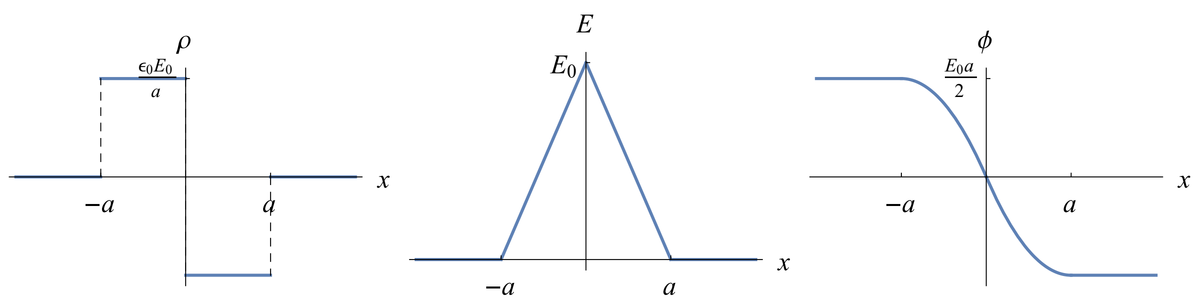
In summary,

$$\phi = \begin{cases} \frac{E_0\,a}{2} & x < -a \\ -E_0\left(\frac{x^2}{2\,a} + x\right) & -a \le x < 0 \\ E_0\left(\frac{x^2}{2\,a} - x\right) & 0 \le x < a \\ -\frac{E_0\,a}{2} & a \le x \end{cases} \tag{6}$$

The charge density $\rho = -\vec{\nabla} \cdot \vec{E} = -\frac{\partial E}{\partial x}$. This is just a straightforward derivative which equals

$$\rho = \begin{cases} 0 & x < -a \\ \frac{\epsilon_0\,E_0}{a} & -a \le x < 0 \\ -\frac{\epsilon_0\,E_0}{a} & 0 \le x < a \\ 0 & a \le x \end{cases} \tag{7}$$

This form of ρ shows us that the charge distribution is two infinity slabs with thickness a and opposite charge densities $\pm \frac{\epsilon_0\,E_0}{a}$ that touch at the $x = 0$ plane. With this, we have a complete picture of the setup.



As a double check, at $x = 0$ the two infinite slabs act effectively like sheets with charge densities $\pm\sigma = \pm\rho\,a$. They create a field pointing to the right with magnitude $\frac{\sigma}{2\,\epsilon_0}$, so the total field is $2\,\frac{\rho\,a}{2\,\epsilon_0} = \frac{\rho\,a}{\epsilon_0}$. Since we found that $\rho = \frac{\epsilon_0\,E_0}{a}$, the field equals E_0, in agreement with the given value. □

Satisfying Laplace

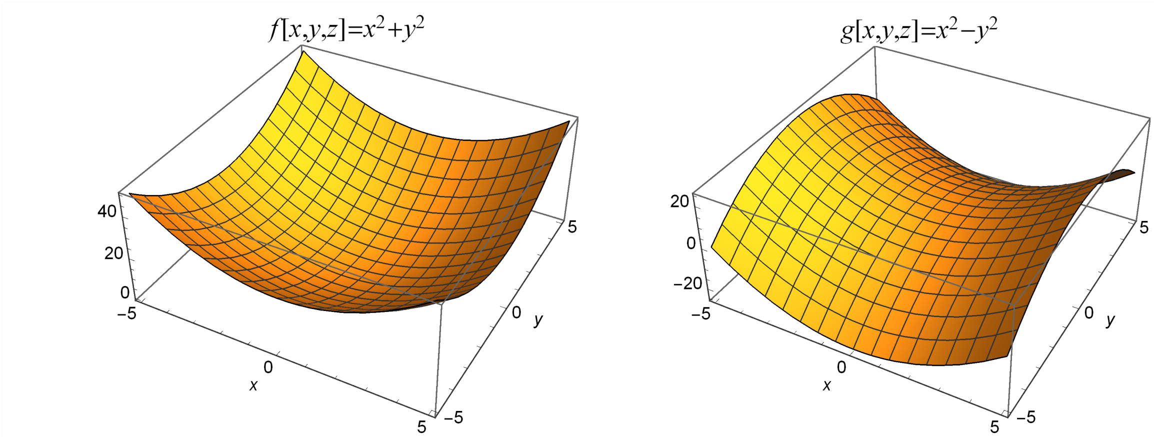
Example

Does the function $f[x, y, z] = x^2 + y^2$ satisfy Laplace's equation? Does the function $g[x, y, z] = x^2 - y^2$?

Suppose $g[x, y, z]$ represented an electric potential and that $g[x, y, z]$ satisfies Laplace's equation. What can you say about the gradient $\vec{\nabla} g$?

Solution

Here is a plot of $f[x, y, z]$ and $g[x, y, z]$.



In Cartesian coordinates, the Laplacian $\nabla^2 = \left\langle \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \right\rangle$. Therefore,

$$\nabla^2 f[x, y, z] = 2 + 2 \neq 0 \tag{8}$$

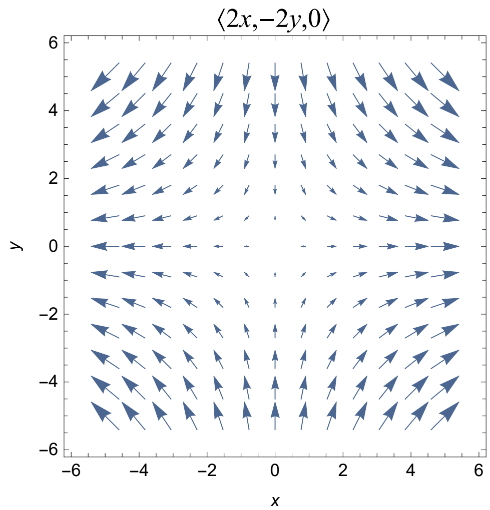
does not satisfy Laplace's equation while

$$\nabla^2 g[x, y, z] = 2 - 2 = 0 \tag{9}$$

does satisfy Laplace's equation. If $g[x, y, z]$ were an electrostatic potential, it would correspond to the electric field

$$-\vec{\nabla} g[x, y, z] = \langle 2\,x, -2\,y, 0 \rangle \tag{10}$$

which is shown below for a constant z,



Because the Laplacian equals zero (or equivalently, the divergence of the gradient of $g[x, y, z]$ equals zero), there is zero net flux out of any closed volume. You should convince yourself that this is true of the figure above. □

Complementary Section: Grounding a Shell

Example

A conducting spherical shell has charge Q and radius R_1. A larger concentric conducting spherical shell has charge $-Q$ and radius R_2.

- What is the potential at all points in space?
- If the outer shell is grounded, explain why nothing happens to the charge on it.
- If instead the inner shell is grounded, find its final charge.

Solution

Before either sphere is grounded, the electric field outside both spherical shells is zero, and hence the outer sphere has the same potential as infinity. This implies that when we ground the outer sphere, nothing will happen. If some negative charge did flow off, then there would be a net positive charge on the two shells, and hence an outward-pointing field for $r > R_2$, which would drag the negative charge back onto the outer shell. Similarly, if some positive charge flowed off, then there would be an inward-pointing field for $r > R_2$ which would drag the positive charge back onto the shell.

If the inner shell is grounded, it must end up with the amount of charge that makes its potential equal to the potential at infinity. Whatever charge distribution ultimately results must satisfy $E = 0$ inside of both spherical conductors. Hence we know that there must be no charge on the *inner* surface of the smaller sphere, and all of the charge Q_1 on the smaller sphere must reside on its *outer* surface. The outer sphere must therefore have a charge $-Q_1$ on its *inner* surface (to satisfy $E = 0$ inside), and hence the remaining charge $Q_1 - Q$ must be equally distributed on the *outer* surface of this larger sphere.

We can now calculate the potential everywhere. First, the potential $\phi[R_2]$ at the surface of the outer sphere must equal

$$\phi[R_2] = \frac{k(Q_1 - Q)}{R_2} \tag{11}$$

since this is identical to the potential from a point charge $Q_1 - Q$ at the origin (using superposition). Next, the potential difference $\phi[R_1] - \phi[R_2]$ between the inner sphere and the outer sphere equals

$$\phi[R_1] - \phi[R_2] = \frac{k\,Q_1}{R_1} - \frac{k\,Q_1}{R_2} \tag{12}$$

which is identical to the potential difference from a point charge Q_1 at the origin (by superposition and the fact that the charge $Q_1 - Q$ on the outer surface of the outer sphere generates no electric field for $r < R_2$). Therefore the potential $\phi[R_1]$ on the surface of the inner sphere, which must be zero at equilibrium, is given by

$$0 = \phi[R_1] = \frac{k(Q_1 - Q)}{R_2} + \frac{k\,Q_1}{R_1} - \frac{k\,Q_1}{R_2} \tag{13}$$

which we can solve to obtain

$$Q_1 = \frac{R_1}{R_2}\,Q \tag{14}$$

Intuitively, if none of the charge leaves (so $Q_1 = Q$), then the inner shell is at a *higher* potential than the outer shell, which in turn is at the same potential as infinity in this case. On the other hand, if all of the charge leaves (so $Q_1 = 0$), then the inner shell is at the same potential as the outer shell, which in turn is at a *lower* potential than infinity in this case. So, by continuity, there must be a value of Q_1 that makes the potential of the inner shell equal to the potential at infinity.

Recommended Problems

This is a list of excellent problems (with solutions) in David Morin's book.

- 3.10 Why leave?

Advanced Section: Image Charges

Mathematica Initialization